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ON THE THEORY OF SUMMABILITY.[*]

BY GLENN JAMES.

Silverman[†] has defined the sum of a divergent series, whose sum to i terms is s_i, to be

$$\text{(I)} \qquad \lim_{x\to\infty} \lim_{n\to\infty} \sum_1^n s_i a_i(x)$$

where, (1) $\lim_{x\to\infty} a_i(x) = 0$, (2) $\sum_1^n |a_i(x)| < K$ or $a_i(x) \geqq 0$, and (3) $\lim_{n=\infty} \sum_1^n a_i(x) = 1$; and he proved that this definition is regular.[‡] This paper considers similar but more general summation processes. A theorem which is fundamental in the theory of summability is developed. From this theorem certain deductions are made. The most important of these are: A test for the equivalence[§] of two summation definitions, the regularity of the most[||] general definition of the type of (I) and a theorem on the sum of certain product series. Incidentally it is proved that no regular definition which employs positive convergence factors can sum properly divergent series. In the latter part of the paper these theorems are transformed so as to apply to definitions based directly upon the *terms* of the series to be summed.

Consider the repeated limit, $\lim_{x\to\infty} \lim_{n\to\infty} \sum_1^n \phi_i(n, x)\alpha_i(n, x)$, the functions $\phi_i(n, x)$ and $\alpha_i(n, x)$ being defined for; i any positive integer, n a positive integer equal to or greater than i, and x any real number. We have the following fundamental theorem.

THEOREM I. *Given:*

$$(a) \qquad \lim_{x\to\infty} \lim_{n\to\infty} \alpha_i(n, x) = 0,$$

i fixed,

$$(b) \qquad \sum_1^n |\alpha_i(n, x)| < K,$$

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[*] Presented to the American Mathematical Society, under a different title, December, 1916.

[†] Silverman, University of Missouri Studies, Vol. 1, No. 1. For other general definitions of summability see: Chapman, Quarterly Journal of Mathematics, Vol. 43, and Smail, Dissertation, Columbia University.

[‡] A definition is said to be *regular* when it sums every convergent series to its ordinary sum.

[§] Two definitions are said to be *equivalent* when every series summed by one is summed by the other to the same sum.

[||] "The most general" in the sense that its assumptions, as will be seen, are necessary as well as sufficient to establish its regularity.

K properly chosen,

(c) $$|\phi_i(n, x)| \leqq e_i,$$

for all values of x and n, where e_i, $i = 1, 2, 3, \cdots$, is a sequence of constants with limit zero. It follows from these assumptions that

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i(n, x)\alpha_i(n, x) = 0.$$

Proof. Evidently we have

$$(1)\quad \left|\sum_1^n \phi_i(n, x)\alpha_i(n, x)\right| \leqq \left|\sum_1^I \phi_i(n, x)\alpha_i(n, x)\right| + \left|\sum_{I+1}^n \phi_i(n, x)\alpha_i(n, x)\right|$$

$$(2)\quad \leqq e\left|\sum_1^I \alpha_i(n, x)\right| + \sum_{I+1}^n |\phi_i(n, x)| \cdot |\alpha_i(n, x)|,$$

where $e \geqq e_i$, $i = 1, 2, 3, \cdots$. Now for an arbitrary ϵ we can choose an I such that

$$(3)\quad |\phi_i(n, x)| \leqq e_i < \epsilon/2K,$$

for every $i > I$. Then with I fixed we can, because of assumption (*a*), choose X and N_x, (N depending on x), such that

$$(4)\quad \left|\sum_1^I \alpha_i(n, x)\right| < \epsilon/2e$$

for $x > X$ and $n > N_x$. Applying (3) and assumption (*b*) to the second term and (4) to the first term in the right member of (2), we obtain

$$\left|\sum_1^n \phi_i(n, x)\alpha_i(n, x)\right| < e(\epsilon/2e) + K(\epsilon/2K) = \epsilon,\ x > X \text{ and } n > N_x,$$

from which it follows that

$$\lim_{x+\infty}\lim_{n\to\infty}\sum_1^n \phi_i(n, x)\alpha_i(n, x) = 0.$$

As a corollary of this theorem we have the following test for the equivalence of two definitions of the sum of a series. Let

$$(A)\quad S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i\beta_i(n, x)$$

and

$$(B)\quad S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i\beta_i'(n, x)$$

be two such definitions, where s_i is the ordinary sum of the series to i terms.

Corollary I. *If (A) sums a series, (B) sums it to the same sum provided the difference, $(s_i\beta_i' - s_i\beta_i)$ can be expressed as the product of two* *factors* α_i *and* ϕ_i *such that* α_i *satisfies* (*a*) *and* (*b*) *and* ϕ_i *satisfies* (*c*) *of Theorem* I.

For we can write

$$\sum_1^n s_i\beta_i' = \sum_1^n (s_i\beta_i + s_i\beta_i' - s_i\beta_i) = \sum_1^n s_i\beta_i + \sum_1^n (s_i\beta_i' - s_i\beta_i)$$

and the repeated limit of the second term on the right is zero by hypothesis.

We will apply this test to show the equivalence of the definitions,

$$S = \lim_{n\to\infty} 1/n \sum_1^n s_i{}^* \tag{1}$$

and

$$S = \lim_{n\to\infty} \sum_1^n s_i \frac{i+k}{i(n+k)}. \tag{2}$$

Here

$$s_i\beta_i' - s_i\beta_i = \pm \frac{s_i(ik - nk)}{ni(n+k)},$$

the sign depending upon whether we set $1/n = \beta_i'$, or $1/n = \beta_i$. Evidently a necessary condition for the existence of either (1) or (2) is

$$\lim_{n\to\infty} s_n/n = 0.$$

From this condition and the fact that $n \geqq i$ it follows that

$$\lim_{i\to\infty} \frac{s_i(ik - nk)}{i(n+k)} = 0.$$

Consequently, if (2) is known to exist, we choose

$$\frac{s_i(ik - sk)}{i(n+k)} = \phi_i \qquad \text{and} \qquad 1/n = \alpha_i,$$

while if (1) is known to exist, we choose

$$-\,s_i/i = \phi_i \qquad \text{and} \qquad \frac{ik - nk}{n(n+k)} = \alpha_i.$$

These forms of α_i and ϕ_i satisfy the assumptions of the corollary.

COROLLARY II. *Given*:

$$\lim_{x\to\infty}\lim_{n\to\infty} \alpha_i(n, x) = 0, \tag{a}$$

$$\sum_1^n |\alpha_i(n, x)| < K, \tag{b}$$

$$\lim_{x\to\infty}\lim_{n\to\infty} \sum_1^n \alpha_i(n, x) = G, \tag{c'}$$

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* Frobenius's Mean, See Bromwich, "Theory of Infinite Series," p. 310.

(*d*) *$\phi_i(n, x)$ is bounded for all values of i, n and x, and the sequence $\phi_i(n, x)$, $i = 1, 2, 3, \cdots$, converges to ϕ_i uniformly in n and x. Then*

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i(n, x)\alpha_i(n, x) = C\cdot\phi.$$

For we can write

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i\alpha_i - C\cdot\phi = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i\alpha_i - \phi\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \alpha_i$$
$$= \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n (\phi_i - \phi)\alpha_i,$$

and the right member of the second identity is zero, since $(\phi_i - \phi)$ satisfies assumption (*c*) of the above theorem.

If $\alpha_i(n, x)$ takes the form $f(z_i)\Delta z_i$ where $f(z)$ is absolutely integrable on the interval a to b and $\sum_1^n \Delta z_i = b - a$, and if ϕ_i is the sum to i terms of a convergent series, the conclusion of the above corollary takes the form

$$\lim_{n\to\infty}\sum_1^n f(z_i)\phi_i\Delta z_i = \lim_{i\to\infty}\phi_i\int_a^b f(z)dz.$$

This summation process is regular whenever the integral on the right is unity. Hence we can obtain any number of regular definitions of summability by choosing different functions with which it is possible to set up definite integrals equal to unity, the functions being absolutely integrable on their respective intervals. For example, from $\int_0^1 dz$ we infer the regularity of Frobenius's Mean, $\lim_{n\to\infty} 1/n\sum_1^n s_i$ and from $\int_0^{\pi/2}\sin z dz$ we infer this property of $\lim_{n\to\infty}\sum_1^n s_i(\pi/2n)\sin i\pi/2n$.

More generally we consider the following definition.

$$(S) \qquad S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i\alpha_i(n, x),$$

where $\alpha_i(n, x)$ is restricted as follows:

$$(a') \qquad \lim_{x\to\infty}\lim_{n\to\infty}\alpha_i(n, x) = 0,$$

$$(b') \qquad \sum_1^n |\alpha_i(n, x)| \leqq K, \qquad x > X, \qquad n > N_x{}^*$$

and

$$(c'') \qquad \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \alpha_i(n, x) = 1.$$

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[*] That is, $\Sigma_1^n\,|\,\alpha_i\,|$ is bounded for x greater than a properly chosen X and n greater than N dependent upon x. This assumption is all that is essential in (*b*) of Theorem I and Corollary II.

That these restrictions are *sufficient* to make (S) regular is an obvious deduction from corollary II. Toeplitz* has proved that a set of restrictions, equivalent to the above when α_i is a function of n only are necessary and sufficient to make the definition $S = \lim_{n\to\infty} \sum_1^n s_i\alpha_i(n)$ regular. His proof of the *necessity* of these restrictions is applicable to the case we are considering, that is, when α_i is a function of both n and x. In this connection it is interesting to consider the necessity of the restrictions of (S) in order to sum a single convergent series to its ordinary sum. We omit the trivial case in which all the terms of the series are zero. It is evident that if (c'') holds, either (a') or (b') or both may fail and the series whose sum to i terms is constant will still sum to that constant. *However if* (a') *and* (b') *hold, and* (S) *sums, to its ordinary sum, a single convergent series whose sum is not zero, then*

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \alpha_i = 1,$$

if it exists. This is an immediate consequence of Corollary II.

An interesting special case of (S) is obtained by assuming that α_i is positive. This assumption and (c'') imply (b'). We then have the definition,

$$(S) \qquad S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i\alpha_i(n, x),$$

where α_i is positive and satisfies (a') and (c'') of (S).

Definition (S) *may sum properly divergent series*† *but* (S') *can not.*

Consider the definition for which the divergence factors are

$$3/n,\ -1/n,\ 3/n,\ -1/n,\ \cdots,\ [1-2(-1)^i]/n,\ \cdots.$$

These forms for α_i satisfy the restrictions of (S), and this definition sums, to zero, the properly divergent series whose sequence of sums is

$$\sqrt{1},\ 3\sqrt{1},\ \sqrt{2},\ 3\sqrt{2},\ \cdots,\ [2+(-1)^i]\sqrt{\frac{2i+1-(-1)^i}{4}},\ \cdots,$$

for

$$\sum_1^n s_i\alpha_i = \left|\begin{array}{l} 0,\ n \text{ even} \\ (3/n)\sqrt{(n+1)/2},\ n \text{ odd} \end{array}\right.$$

The fact that (S') can not sum properly divergent series is an instance of the following general theorem.

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* Toeplitz, Prace Matematyczno-fizyczno, Vol. 22 (1911), p. 113.

† A properly divergent series is one for which $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \Sigma\, u_i = +\infty$ or $-\infty$. In theorem II, which follows, it is only necessary to consider the case in which $\lim_{n\to\infty} s_n = +\infty$.

Let α_i be not only a function of i, a positive integer, but of any other variables whatever and let $\lim{}'\text{s} \sum_1^n s_i\alpha_i$ denote the result of applying any limiting processes whatever to $\sum_1^n s_i\alpha_i$. Then we have

THEOREM II. *If the definition,*

$$S = \lim{}'\text{s} \sum_1^n s_i\alpha_i,$$

where $\alpha_1 \geqq 0$, *is regular, it can not sum properly divergent series.*

PROOF. Let the series whose sum to i terms is s_i properly diverge. Then for an arbitrary C, we can choose j such that $S_i \geqq C$ for every $i \geqq j$. Now consider the sequences,

$$(1) \qquad s_1,\ s_2,\ s_3,\ \cdots,\ s_j,\ s_{j+1},\ s_{j+2},\ \cdots,$$

$$(2) \qquad s_1,\ s_2,\ s_3,\ \cdots,\ s_{j-1},\ C,\ C,\ C,\ \cdots,$$

$$(3) \qquad 0,\ 0,\ 0,\ \cdots,\ s_j - C,\ s_{j+1} - C,\ s_{j+2} - C,\ \cdots.$$

The second sequence arises from a series that is summable to C. Consequently if the series from which the first series arises, that is the given series, is summable, we have from elementary limit theory

$$\lim{}'\text{s} \sum_1^n s_i\alpha_i - C = \lim{}'\text{s} \sum_j^n (s_i - C)\alpha_i.$$

Since the right member is positive however large we choose C, we conclude that $\lim{}'\text{s} \sum_1^n s_i\alpha_i$ does not exist.

As a direct application of Theorem I, we have the following theorem, which is not without interest for its own sake.

THEOREM III. *If the series whose sum to* i *terms is* s_i' *converges or oscillates between finite limits and is summable to* s', *by* (S), *and if the series whose sum to* i *terms is* s_i *converges to* s, *then the series whose sum to* i *terms is* $s_i's_i$ *sums to* ss', *by* (S).

PROOF. Since s_i' is bounded, assumptions (a) and (b) of Theorem I are satisfied by $s_i'\alpha_i$ whenever they are satisfied by α_i. Hence, in that theorem, we may replace α_i by $s_i'\alpha_i$ and ϕ_i by $(s_i - s)$ and obtain

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_1^n (s_i - s)s_i'\alpha_i = 0$$

Whence

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_{1}^{n} s_i s_i' \alpha_i = \lim_{x\to\infty}\lim_{n\to\infty}\sum_{1}^{n} s s_i' \alpha_i$$

$$= s \lim_{x\to\infty}\lim_{n\to\infty}\sum_{1}^{n} s_i' \alpha_i$$

$$= s s_i'.$$

Since many special definitions are of the form

$$(a) \qquad S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_{1}^{n} u_i \delta_i(n, x),$$

where u_i is the ith term of the series to be summed, it is desirable to transform Theorem I into a form that is directly applicable to such definitions.

To accomplish this, substitute,[*] in Theorem I,

$$(1) \qquad \phi_i = \sum_{1}^{i} \theta_h$$

and

$$(2) \qquad \alpha_i = \left| \begin{array}{ll} \delta_i - \delta_{i+1}, & i < n \\ \delta_n, & i = n \end{array} \right.$$

and collect the coefficients of θ_1, θ_2, $\cdots$, in the conclusion. This gives

THEOREM I. *Given:*

$$(e) \qquad \lim_{x\to\infty}\lim_{n\to\infty} (\delta_i - \delta_{i+1}) = 0,$$

i fixed,

$$(f) \qquad \left[\sum_{1}^{n-1} |\delta_i - \delta_{i+1}| + |\delta_n| \right] < K,$$

or

$$\sum_{1}^{n-1} |\delta_i - \delta_{i+1}| < K'$$

and $|\delta_n| < K''$, *and*

$$(g) \qquad \left| \sum_{1}^{n} \theta_i \right| \leqq e_i,$$

where e_i, $i = 1, 2, 3, \cdots$, *is a sequence of constants with limit zero, it follows from these hypotheses that*

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_{1}^{n} \theta_i(n, x) \delta_i(n, x) = 0.$$

From (2) we see that $\delta_1 = \sum_{1}^{n} \alpha_i$. Hence ($c'$) of Corollary II gives us

$$\lim_{x\to\infty}\lim_{n\to\infty} \delta_1 \doteq C$$

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[*] This substitution is suggested by collecting the coefficients $u_1, u_2, u_3, \ldots$, in $S_n = \Sigma_1^n (\Sigma_1^i u_h) \alpha_i$. The coefficient of u_i is $\Sigma_1^n \alpha_h$ ($= \delta_i$).

which taken with (e) and (f) give (e') and (f') of the following transform of Corollary II.

COROLLARY II′. *Given:*

$$(e') \qquad \lim_{x\to\infty}\lim_{n\to\infty}\delta_i = C,$$

i fixed.

$$(f') \qquad \sum_1^{n-1}|\delta_i - \delta_{i+1}| < K, \qquad x > X, \qquad n > N_x,$$

and

$$(g') \qquad \sum_1^{i}\theta_h(n, x)$$

is bounded for all values of i, n and x, and

$$\sum_1^{i}\theta_h(n, x)$$

converges to θ uniformly in n and x, then

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n\theta_i\delta_i = C\cdot\theta.$$

Consider now the definition

$$(u) \qquad S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n u_i\delta_i,$$

where

$$(e'') \qquad \lim_{x\to\infty}\lim_{n\to\infty}\delta_i = 1,$$

$$(f'') \qquad |\delta_i - \delta_{i+1}| < K,$$

$x > X$ and $n > N_x$, and the definition, (u'), obtained by replacing (f'') by the assumption

$$(f''')\dagger \qquad \delta_1, \delta_2, \delta_3, \cdots,$$

is a positive, decreasing sequence. These are the definitions into which (S) and (S'), respectively, transform. Hence they possess all the properties that the latter possess. For instance, the restrictions of (u) are *necessary and sufficient* to make it regular. A similar transformation of Theorem II is easily made and is not without interest.

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[*] Similarly we could have used $\lim_{x\to\infty}\lim_{n\to\infty}\Sigma_j^n\alpha_i = C$, i fixed, instead of (a) and (c') of Corollar II, p. 4.

[†] L. L. Smail discusses this definition in these Annals, Vol. 20 (1918).